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Negative-dimensional groups in quantum physics

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Abstract. We show that an explicit definition of negative-dimensional classical groups in terms of their action on Grassmann representation spaces, together with Weyl's character formula for the Grassmann tensorial representations of such groups, leads to a framework in which some surprising 'negative-dimensional' properties of group theoretic invariants arise naturally. As an application, we show that the spectra of the Grassmann harmonic oscillator and of Grassmann angular momentum are related to their bosonic counterparts by a simple analytic continuation in the dimension.

1. Introduction

Physicists have various good reasons to believe that Grassmann variables are (in some appropriate sense) 'negative dimensional'. This belief is usually based on their (Berezin) integration properties (Parisi and Sourlas 1979, Dunne and Halliday 1987, 1988). In this paper we show that the correspondence between Grassmanns and negative dimensions goes far deeper, and has interesting consequences in the representation theory of the classical groups, and in quantum spectral analysis.

One motivation for this work is the fact that the quantisation of Grassmann systems has come to play a vital role in our current understanding of quantisation, particularly of constrained quantisation. Grassmann phase space variables, corresponding to the constrained degrees of freedom, feature prominently both in the path integral formulation of Fradkin and Vilkovsky (1975), and in the superfield formulation of Bonora *et al* (1981). This coincides quite nicely with the idea of constraints as being 'removed dimensions', a fact that has been used by Aratyn *et al* (1987) in their incorporation of Parisi–Sourlas symmetry into the BRST quantisation of reparametrisation-invariant theories. Grassmann variables have also been used as a form of 'dimensional reduction' in a path integral approach to stochastic quantisation (McClain *et al* 1982, Gozzi 1983).

In § 2, we discuss the tensorial representation theory of the classical groups on a Grassmann space, and show that the irreducible characters for such representations are related to the characters of bosonic tensor representations in a simple way. An explicit realisation of this relation is given in the appendix.

In § 3, we summarise various curious mathematical results of Penrose (1971), King (1971) and Čvitanović and Kennedy (1982) concerning the analytic continuation of group invariants to negative dimensions. We then *define* the negative-dimensional classical groups in terms of their action on a Grassmann space, and in such a way

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that, with these definitions, the aforementioned curious results concerning 'negative-dimensional' group invariants follow from the results of § 2. We show in § 4 that this construction can be extended to include the spinor representations of SO$(-n)$, which we find to be infinite-dimensional 'symplectic spinor' representations of Sp(n).

In §§ 5 and 6, we apply negative-dimensional group theory to the archetypal quantum spectral problems for the classical groups—the d-dimensional harmonic oscillator for SU(d) and d-dimensional angular momentum for SO(d). Finally, we conclude with some discussion concerning further applications and generalisations of the ideas presented in this paper.

2. Grassmann tensorial representation theory

In Klein's Erlangen programme and Weyl's theory of invariants (Weyl 1939) the classical groups are defined in terms of transformations on some n-dimensional vector space V. For example, GL(n) is defined to be the group of all non-singular $n\times n$ matrices, and its defining representation gives its action as linear transformations on an n-dimensional Euclidean space ε_n:

$$A\in \mathrm{GL}(n) \qquad \begin{aligned} &x\to x'=Ax \\ &x^i\to x'^i=A^i{}_j x^j. \end{aligned} \tag{2.1}$$

The 'tensor' representations are then constructed by considering the action of GL(n) on the rank-r tensor space $\varepsilon_n\otimes\varepsilon_n\otimes\ldots\otimes\varepsilon_n\equiv(\varepsilon_n)^{\otimes r}$. The special tensors

$$x^{i_1}_{(1)}x^{i_2}_{(2)}\ldots x^{i_r}_{(r)} \tag{2.2}$$

formed from products of the components of r n-vectors $x_{(1)}, x_{(2)},\ldots,x_{(r)}$ span the tensor space, and furthermore they carry an $(n^r\times n^r)$-dimensional (reducible) representation of GL(n):

$$A\in \mathrm{GL}(n)\colon x^{i_1}_{(1)}x^{i_2}_{(2)}\ldots x^{i_r}_{(r)}\mapsto A^{i_1}{}_{j_1}A^{i_2}{}_{j_2}\ldots A^{i_r}{}_{j_r}x^{j_1}_{(1)}x^{j_2}_{(2)}\ldots x^{j_r}_{(r)}. \tag{2.3}$$

Now there is also an (independent) action of the symmetric group S_r on the rank-r tensor space which permutes the elements of the different copies of ε_n in the tensor product $(\varepsilon_n)^{\otimes r}$:

$$p\in S_r\colon x^{i_1}_{(1)}x^{i_2}_{(2)}\ldots x^{i_r}_{(r)}\mapsto x^{i_1}_{(p_1)}x^{i_2}_{(p_2)}\ldots x^{i_r}_{(p_r)}=x^{i_{p_1}}_{(1)}x^{i_{p_2}}_{(2)}\ldots x^{i_{p_r}}_{(r)}. \tag{2.4}$$

As these actions (2.3) and (2.4) commute, we can use the decomposition of the tensor space into irreducible components under S_r to decompose the tensor space into irreducible components under GL(n). This decomposition is achieved using Young tableaux (Hammermesh 1962, Boerner 1963, Littlewood 1950) and in particular the *Young symmetriser* e_λ corresponding to the Young tableau λ ($\lambda\equiv(\lambda_1,\lambda_2,\ldots,\lambda_k)$ is a partition of r, giving the number λ_i of boxes in the ith row of the Young tableau):

$$e_\lambda=\sum_{h,v}(-1)^{v_\lambda}h_\lambda v_\lambda \tag{2.5}$$

where h and v refer to the horizontal and vertical permutations of the Young tableau box entries along the rows and columns, respectively. The action of the symmetrisers e_λ for all normal Young tableaux on the basis tensors (2.2) generates all the inequivalent irreducible invariant subspaces of the rank-r tensor space.

Now this whole procedure can be followed through taking the n-dimensional vector space carrying the defining representation to be an n-dimensional Grassmann space G_n. (This is, of course, part of the representation theory of the Lie supergroups (Scheunert 1979, Balantekin and Bars 1981, Dondi and Jarvis 1981)). Once again, the rank-r tensor space is spanned by the special tensors

$$\Theta^{i_1}_{(1)}\Theta^{i_2}_{(2)}\ldots\Theta^{i_r}_{(r)} \tag{2.6}$$

and these carry an $(n^r\times n^r)$-dimensional (reducible) representation of $\mathrm{GL}(n)$:

$$A\in \mathrm{GL}(n)\colon \Theta^{i_1}_{(1)}\Theta^{i_2}_{(2)}\ldots\Theta^{i_r}_{(r)}\mapsto A^{i_1}{}_{j_1}A^{i_2}{}_{j_2}\ldots A^{i_r}{}_{j_r}\Theta^{j_1}_{(1)}\Theta^{j_2}_{(2)}\ldots\Theta^{j_r}_{(r)}. \tag{2.7}$$

The action of the symmetric group S_r on the rank-r tensor space is

$$p\in S_r\colon \Theta^{i_1}_{(1)}\Theta^{i_2}_{(2)}\ldots\Theta^{i_r}_{(r)}\mapsto\Theta^{i_1}_{(p_1)}\Theta^{i_2}_{(p_2)}\ldots\Theta^{i_r}_{(p_r)}=(-1)^p\Theta^{i_{p_1}}_{(1)}\Theta^{i_{p_2}}_{(2)}\ldots\Theta^{i_{p_r}}_{(r)}. \tag{2.8}$$

Thus, the Grassmann (anticommuting) nature of the Θ means that when we write the action of the symmetric group S_r on the tensor space as a permutation of the *indices* of the basis tensors (2.6), we must include a $(-1)^p$ factor. Hence, the Young symmetriser e_λ acts on (2.6) in exactly the same way as

$$\sum_{h,v}(-1)^{h_\lambda}h_\lambda v_\lambda \tag{2.9}$$

acts on (2.2), and so instead of symmetrising over rows and antisymmetrising over columns, we are symmetrising over columns and antisymmetrising over rows. Thus, (2.9) corresponds to the Young symmetriser for the 'transposed' (rows and columns interchanged, i.e. reflected about the leading diagonal) Young tableau $\tilde\lambda$. This is, in fact, the *only* difference from the conventional case of a bosonic (commuting) tensor space, so the reduction of the tensor representations of $\mathrm{GL}(n)$ into irreducible components proceeds as usual, with $\tilde\lambda$ playing the role of λ.

To see how this affects the irreducible characters, we use the Frobenius formula which relates to the characters φ_λ of $\mathrm{GL}(n)$ in the irreducible representation λ to the characters χ_λ of S_r in the irreducible representation λ (see, for example, ch VI of Boerner (1963)):

$$\varphi_\lambda(A)=\sum_{(\alpha)}\frac{\chi_\lambda(A)}{\alpha_1!\ldots\alpha_r!}\left(\frac{\sigma_1}{1}\right)^{\alpha_1}\left(\frac{\sigma_2}{2}\right)^{\alpha_2}\cdots\left(\frac{\sigma_r}{r}\right)^{\alpha_r} \tag{2.10}$$

where

$$\sigma_k\equiv\mathrm{Tr}(A^k)$$

$$\alpha\equiv\text{permutation class }(\alpha_1+2\alpha_2+\ldots+r\alpha_r=r).$$

From (2.8) it follows that the only change when we derive this formula on a Grassmann representation space is that each permutation class α in the sum picks up a minus sign if it is odd. So

$$\varphi_\lambda(A;\text{Grassmann})=\sum_{(\alpha)}\frac{(-1)^\alpha\chi_\lambda(A)}{\alpha_1!\ldots\alpha_r!}\left(\frac{\sigma_1}{1}\right)^{\alpha_1}\left(\frac{\sigma_2}{2}\right)^{\alpha_2}\cdots\left(\frac{\sigma_r}{r}\right)^{\alpha_r}. \tag{2.11}$$

Now it is a result from the theory of the characters of the symmetric group that (Littlewood 1950)

$$\chi_{\tilde\lambda}(\alpha)=(-1)^\alpha\chi_\lambda(\alpha). \tag{2.12}$$

Combining with (2.10) and (2.11) we obtain the result

Lemma 1. For $A \in \mathrm{GL}(n)$,

$$\varphi_\lambda(A; \text{Grassmann}) = \varphi_{\tilde{\lambda}}(A; \text{boson}). \tag{2.13}$$

For an explicit expression of this result in terms of Weyl's character formula, see the appendix.

Now consider the irreducible tensor representations of the important classical subgroups of $\mathrm{GL}(n)$: $\mathrm{U}(n)$, $\mathrm{SU}(n)$, $\mathrm{O}(n)$, $\mathrm{SO}(n)$ and $\mathrm{Sp}(n)$. The conditions defining these subgroups may be expressed as matrix conditions independent of the commuting or anticommuting nature of the tensor space. (However, note that the way in which we realise these conditions as invariant bilinear forms on the tensor space does depend on the nature of the tensor space—see § 3.) Hence, the further reduction of the $\mathrm{GL}(n)$ irreducible representations when we restrict to these subgroups is the same for the bosonic and Grassmann tensor space. So we obtain lemma 2.

Lemma 2. If G is any one of the classical groups $\mathrm{U}(n)$, $\mathrm{SU}(n)$, $\mathrm{O}(n)$, $\mathrm{SO}(n)$ or $\mathrm{Sp}(n)$ and $A \in \mathrm{G}$, then

$$\varphi_\lambda^{\mathrm{G}}(A; \text{Grassmann}) = \varphi_{\tilde{\lambda}}^{\mathrm{G}}(A; \text{boson}). \tag{2.14}$$

3. Negative dimensional classical groups

There have been various 'sightings' of 'negative dimensions' in the context of Lie group theory over the years. Penrose (1971) noted, in the context of a diagrammatic method for computing algebraic invariants, that certain angular momentum invariants could be computed by representing $\mathrm{SU}(2) \simeq \mathrm{Sp}(2)$ as '$\mathrm{SO}(-2)$'. (This observation was based on Penrose's graphical method for calculating algebraic invariants, in which the computation can be converted into a colouring problem which permitted solution when the number of colours was 2 or -2.) Unfortunately, no explicit construction of these 'binors' was given. King (1971), in giving simple definitive formulae for the dimensions of the irreducible representations of the classical groups $\mathrm{SU}(n)$, $\mathrm{SO}(n)$ and $\mathrm{Sp}(n)$, noticed a peculiar symmetry of these formulae. If λ_s is a Young tableau with s boxes, and if the dimensions of the corresponding irreducible representations of $\mathrm{SU}(n)$, $\mathrm{SO}(n)$ and $\mathrm{Sp}(n)$ are denoted by $D\{\lambda_s : n\}$, $D[\lambda_s; n]$ and $D\langle\lambda_s; n\rangle$, respectively, then King noticed that

$$D\{\lambda_s; n\} = (-1)^s D\{\tilde{\lambda}_s; -n\} \tag{3.1}$$

$$D[\lambda_s; n] = (-1)^s D\langle\tilde{\lambda}_s; -n\rangle. \tag{3.2}$$

Once again, no explanations or interpretation was offered for these unusual relations.

Čvitanović and Kennedy (1983) used Penrose's diagrammatic ('birdtrack') notation to prove the following important results.

(i) For all $3m-j$ coefficients constructed from tensor representations of $\mathrm{SU}(n)$, the interchange of symmetrisation and antisymmetrisation is equivalent (up to an overall sign of the $3m-j$ coefficient) to the 'analytic continuation' $n \to -n$.

(ii) For all $3m-j$ coefficients constructed from tensor representations of SO(n) and Sp(n), the interchange of symmetrisation and antisymmetrisation, together with the interchange of SO(n) and Sp(n), is equivalent (up to an overall sign of the $3m-j$ coefficient) to the 'analytic continuation' $n \to -n$.

Čvitanović and Kennedy expressed these results symbolically as

$$\mathrm{SU}(-n) \cong \overline{\mathrm{SU}(n)} \tag{3.3a}$$

$$\mathrm{SO}(-n) \cong \overline{\mathrm{Sp}(n)} \tag{3.3b}$$

$$\mathrm{Sp}(-n) \cong \overline{\mathrm{SO}(n)} \tag{3.3c}$$

where the overbar means symmetrisation and antisymmetrisation are interchanged.

We can, in fact, give explicit concrete evidence for such relations by looking at the closed-form expressions for the independent generalised Casimirs of the classical groups (see, for example, ch 9 of Barut and Raczka (1986)) in the totally symmetric and totally antisymmetric representations (these are often the most important irreducible representations for physical applications). For U(n), the pth-order generalised Casimir in the totally symmetric rank-r tensor representation is

$$C_p^{\mathrm{U}(n)}(r, 0, \ldots, 0) = r(r+n-1)^{p-1} \tag{3.4a}$$

while in the totally antisymmetric rank-r tensor representation

$$C_p^{\mathrm{U}(n)}(1, 1, \ldots, 1) = r(1+n-r)^{p-1}. \tag{3.4b}$$

For SU(n), the corresponding formulae are

$$C_p^{\mathrm{SU}(n)}(r, 0, \ldots, 0) = \frac{r(n+r)(n-1)}{n(n+r-1)}\left\{\left[\frac{(n+r)(n-1)}{n}\right]^{p-1} - \left[\frac{-r}{n}\right]^{p-1}\right\} \tag{3.5a}$$

and

$$C_p^{\mathrm{SU}(n)}(1, 1, \ldots, 1) = \frac{r(n+1)(n-r)}{n(n+1-r)}\left\{\left[\frac{(n+1)(n-r)}{n}\right]^{p-1} - \left[\frac{-r}{n}\right]^{p-1}\right\}. \tag{3.5b}$$

There is a remarkable symmetry in these formulae (not commented on by Barut and Raczka):

$$C_p^{\mathrm{U}(n)}(1, 1, \ldots, 1) = (-1)^{p-1} C_p^{\mathrm{U}(m)}(r, 0, \ldots, 0)|_{m=-n} \tag{3.6a}$$

$$C_p^{\mathrm{SU}(n)}(1, 1, \ldots, 1) = (-1)^{p-1} C_p^{\mathrm{SU}(m)}(r, 0, \ldots, 0)|_{m=-n}. \tag{3.6b}$$

For the orthogonal and symplectic groups, the results are even more striking. For SO($2n$) and Sp($2n$) we find

$$\begin{aligned} C_p(r, 0, \ldots, 0) &= (r+2\alpha)^p + (-r)^p + (2\alpha+\beta-1) + \frac{\alpha(\beta+1)}{2(\alpha-1)}\,\frac{(r+2\alpha)^p - (-r)^p}{r+\alpha} \\ &\quad + (2\alpha-1)\left(1+\frac{\beta+1}{2(\alpha-1)}\right)\left(\frac{-1+(-r)^{p-1}}{r+1} - \frac{(r+2\alpha)^p - 1}{r+2\alpha-1}\right) \end{aligned} \tag{3.7a}$$

and

$$\begin{aligned} C_p(1, 1, \ldots, 1) &= -(2\alpha+2-r)^p - r^p + (-1)^p(2\alpha+\beta+3) + \frac{(\alpha+1)(\beta-1)}{2(\alpha+2)}\,\frac{r^p - (2\alpha+2-r)^p}{\alpha+1-r} \\ &\quad + (2\alpha+3)\left(1+\frac{\beta-1}{2(\alpha+2)}\right)\left(\frac{r^p-(-1)^p}{r+1} + \frac{(2\alpha+2-r)^p - (-1)^p}{2\alpha+3-r}\right) \end{aligned} \tag{3.7b}$$

where

$$\alpha=\begin{cases} n-1 & \text{for } \mathrm{SO}(2n) \\ n & \text{for } \mathrm{Sp}(2n) \end{cases}$$

$$\beta=\begin{cases} 1 & \text{for } \mathrm{SO}(2n) \\ -1 & \text{for } \mathrm{Sp}(2n). \end{cases}$$

Remarkably, we find

$$C_p^{\mathrm{SO}(2n)}(1,1,\ldots,1)=(-1)^{p-1}C_p^{\mathrm{Sp}(2m)}(r,0,\ldots,0)|_{m=-n} \tag{3.8a}$$

and

$$C_p^{\mathrm{Sp}(2n)}(1,1,\ldots,1)=(-1)^{p-1}C_p^{\mathrm{SO}(2m)}(r,0,\ldots,0)|_{m=-n}. \tag{3.8b}$$

These identities (3.6) and (3.8) give explicit confirmation of the relations (3.3).

The examples (3.1), (3.2), (3.3), (3.6), and (3.8) are all of a very suggestive form when viewed in the light of the result (2.14) relating the characters of bosonic tensor representations and Grassmann tensor representations. Note that one can, of course, derive all interesting group invariants (dimensions, $3m-j$ couplings, Casimirs, etc) from the irreducible characters. We are thus motivated to make the following constructive definitions for negative-dimensional classical groups (here we take the Klein–Weyl view that the groups can be defined by their defining representations; the connection with the Lie–Cartan infinitesimal viewpoint is discussed in the subsequent sections.)

$\mathrm{GL}(-n)$ is defined to be the group of non-singular $n\times n$ matrices acting on an n-dimensional Grassmann vector space G_n as in (2.7). Then from (2.13) we conclude

$$\mathrm{GL}(-n)\cong\overline{\mathrm{GL}(n)} \tag{3.9}$$

(the symbol $\cong$ is to be understood in the sense that the irreducible characters are complete, i.e. they encode all the group theoretic information for all the finite-dimensional tensor representations). $\mathrm{U}(-n)$ and $\mathrm{SU}(-n)$ are defined to be restrictions of $\mathrm{GL}(-n)$ in which a Hermitian metric $\delta^{\dot a}_{\dot b}$ is invariant under the actions $\{A\}$ and $\{A^*\}$ of $\mathrm{GL}(-n)$ on G_n and its conjugate G_n^*, respectively (see, for example, Hammermesh 1962):

$$\text{on } G_n: \qquad \Theta^a\mapsto A^a{}_b\Theta^b$$

$$\text{on } G_n^*: \qquad \Theta_{\dot a}\mapsto (A^*)_{\dot a}{}^{\dot b}\Theta_{\dot b}=\Theta_{\dot b}(A^{*\mathrm{T}})^{\dot b}{}_{\dot a}.$$

For $\mathrm{SU}(-n)$ we further require the elements to satisfy $\det A=1$. Then from (2.14) we have

$$\mathrm{U}(-n)\cong\overline{\mathrm{U}(n)} \tag{3.10}$$

$$\mathrm{SU}(-n)\cong\overline{\mathrm{SU}(n)}. \tag{3.11}$$

To define the negative-dimensional orthogonal group we notice that in the Grassmann tensor space the role of the usual symmetric bilinear form δ_{ij} is played by the antisymmetric bilinear form ε_{ij}. ε_{ij} is used to relate G_n to its dual (i.e. to lower indices) and is used to form the analogue of the bosonic inner product $x^2\equiv x^i\delta_{ij}x^j$, namely $\mathbf{\Theta}^2\equiv\Theta^i\varepsilon_{ij}\Theta^j$. (Note that it is *this* $\mathbf{\Theta}^2$ which appears in the Grassmann version of multidimensional Gaussian integration for which the sign of the dimension is reversed—see Dunne and Halliday (1987).) So we define $SO(-n)$ to be the subgroup of unimodular elements of $GL(-n)$ which leave the bilinear form Θ^2 invariant. As these are, in fact, what we would usually call symplectic transformations, we have from (2.14)

$$SO(-n) \cong \overline{Sp(n)} \tag{3.12}$$

(note that n must, of course, be even).

Similarly, the analogue of the bosonic symplectic product $[xy] \equiv x^i \varepsilon_{ij} y^j$ in the Grassmann space is $[\Theta\Pi] \equiv \Theta^i \delta_{ij} \Pi^j$. So we define $Sp(-n)$ to be the subgroup of $GL(-n)$ which leaves invariant bilinear form $[\Theta\Pi]$. As these are what we usually call orthogonal transformations, we have from (2.14)

$$Sp(-n) \cong \overline{SO(n)} \tag{3.13}$$

(note that here n can be odd or even).

Given these constructive *definitions* of the negative-dimensional classical groups, the relations (3.1), (3.2), (3.3), (3.6), and (3.8) all follow from (2.14) (apart from an overall $\pm$ sign). This overall ± 1 factor can be understood by recalling that a character is a *trace*, which we can write as

$$\mathrm{Tr}_\lambda(A) = \sum_{\substack{\text{states} \\ \text{in } \lambda}} \langle \text{state}|A|\text{state}\rangle.$$

In the bosonic case these states can be properly normalised to all have norm $+1$, but in the Grassmann case the natural inner product is indefinite, which means that for some irreducible representations all the states will have norm -1. An explicit example of how this ± 1 factor appears is in the spectral analysis of the negative-dimensional harmonic oscillator (see § 5).

These definitions of negative-dimensional classical groups can be viewed in two ways. Firstly, given the 'data' (3.1), (3.2), (3.3), (3.6) and (3.8), the relation (2.14) provides very convincing evidence that the interpretation of Grassmann spaces as 'negative-dimensional' bosonic spaces is indeed very fundamental, and goes far beyond the original notion based on their integration properties (Parisi and Sourlas 1979, Dunne and Halliday 1987, 1988). Alternatively, given some physically motivated intuition that Grassmanns are negative dimensional, we see that we can construct in a perfectly straightforward manner a framework in which the somewhat bizarre algebraic identities (3.1), (3.2), (3.3), (3.6) and (3.8) arise naturally.

The spectral theory of the classical groups is often formulated in terms of functional analysis (we are usually interested in totally symmetric bosonic representations which correspond to functions of commuting variables). This is an infinite-dimensional form of representation theory which relies heavily on the *field* nature of the space variables (usually $\mathbb{R}^n$ or $\mathbb{C}^n$). Now for the negative-dimensional classical groups, the spectral theory becomes *finite* dimensional as the Grassmann representation space is finite dimensional. Furthermore, the Grassmann space G_n only has a *ring* structure. In these terms it seems surprising at first that the Grassmann representation space is 'sufficient' to match the bosonic representation space, and indeed that a simple analytic continuation $n \to -n$ should connect a finite-dimensional algebraic structure with infinite-dimensional functional analysis.

In §§ 5 and 6 we will see explicitly how all this works by looking at the archetypal quantum spectral problems involving the classical groups: the harmonic oscillator for $SU(n)$ and angular momentum for $SO(n)$.

4. Symplectic or negative-dimensional spinors

The discussion thus far has been concerned with all the finite-dimensional irreducible tensor representations λ associated with Young tableaux. A natural question to ask is: is there a negative-dimensional analogue of the *spinor* representations of the orthogonal group? To arrive at spinors in a language akin to that of the tensorial representation theory, we follow the methods of Dirac (1928) and Brauer and Weyl (1935).

Thierry-Mieg (1984)† has, in fact, shown that the metaplectic representation of $\mathrm{Sp}(2k)$, represented on the space $C(z)$ of functions of the complex variables $z^1, \ldots, z^k$, resembles a negative-dimensional version of the spinor representation of $\mathrm{SO}(2k)$ represented on the space $C(\boldsymbol{\Theta})$ of functions (polynomials) of the Grassmann variables $\Theta^1, \ldots, \Theta^k$. Here we would like to take a different (but complementary) approach, in keeping with the discussion of the preceding sections, and describe spinors using tensorial representation theory.

Dirac considered writing the orthogonal norm as the square of a linear form:

$$(x^1)^2+(x^2)^2+\ldots+(x^n)^2=(\gamma^1x^1+\gamma^2x^2+\ldots+\gamma^nx^n)^2. \tag{4.1}$$

This requires the γ^i to satisfy the anticommutation relations which define a Clifford algebra C_n:

$$\{\gamma^i, \gamma^j\}=2\delta^{ij}. \tag{4.2}$$

Note that there exists a (unique up to equivalence) $2^{[n/2]}\times 2^{[n/2]}$ matrix representation of C_n (Weyl 1939). Now consider a matrix A in the defining representation of the orthogonal group $\mathrm{O}(n)$ acting on the Euclidean vector $\boldsymbol{x}$ as

$$x^i \to x'^i = A^i{}_j x^j. \tag{4.3}$$

Following Brauer and Weyl, we apply the same transformation to the Clifford algebra generators γ^i in the $2^{[n/2]}\times 2^{[n/2]}$ matrix representation

$$\gamma^i \to \gamma'^i = A^i{}_j \gamma^j. \tag{4.4}$$

Then the γ^i also satisfy the Clifford algebra relations (4.2). But since the matrix representation is unique, we must have

$$\gamma'^i = \Delta(A)\gamma^i\Delta(A)^{-1} \tag{4.5}$$

where Δ is a $2^{[n/2]}\times 2^{[n/2]}$ matrix depending on the orthogonal matrix A. In fact, the Δ form a double-valued unitary representation of $\mathrm{O}(n)$, which we call the spinor representation. (Note that, when we restrict A to be an $\mathrm{SO}(n)$ matrix, the situation depends on whether n is even or odd: for odd n there is no change, but for even n the spinor representation Δ splits into two inequivalent representations, Δ_+ and Δ_-, each of degree $2^{[n/2]-1}$.)

Now we cannot follow this procedure on a Grassmann tensor space because the orthogonal 'norm' of a Grassmann vector vanishes identically: $\Theta^i\delta_{ij}\Theta^j \equiv 0$. However, on the Grassmann space it is more natural to consider the *symplectic* form $\boldsymbol{\Theta}^2 \equiv \Theta^i\varepsilon_{ij}\Theta^j$. Then, in the spirit of Dirac, we seek algebraic quantities β^i such that

$$\boldsymbol{\Theta}^2 = (\beta^i\varepsilon_{ij}\Theta^j)^2 \tag{4.6}$$

† The author would like to thank one of the referees for bringing this reference to his attention.

i.e.

$$2(\Theta^1\Theta^2+\Theta^3\Theta^4+\ldots+\Theta^{n-1}\Theta^n)=(\beta^1\Theta^2-\beta^2\Theta^1+\ldots+\beta^{n-1}\Theta^n-\beta^n\Theta^{n-1})^2.$$

(Notice that n is required to be even.) This requires the β^i to satisfy the *commutation* relations

$$[\beta^i,\beta^j]=2\varepsilon^{ij}. \tag{4.7}$$

These commutation relations can be represented as the commutation relations of the Heisenberg algebra for n independent sets of bosonic creation and annihilation operators:

$$[z^i,z^j]=2\varepsilon^{ij}$$

where

$$z=\sqrt{2}(a^{1+},a^1,a^{2+},a^2,\ldots,a^{n+},a^n). \tag{4.8}$$

Thus we can represent the β^i by a unitary irreducible infinite-dimensional representation, as is usually done in terms of differential operators on Hilbert space in n-dimensional quantum mechanics.

Now, consider making linear transformations analogous to (4.3) and (4.4) but now with A being a matrix in the defining representation of Sp(n):

$$\Theta^i\to\Theta'^j=A^i{}_j\Theta^j$$

and

$$\beta^i\to\beta'^i=A^i{}_j\beta^j$$

where the β^i are represented as in (4.8). Then the β'^i also satisfy the commutation relations (4.7). But the Stone–Von Neumann theorem (see Abraham and Marsden 1978, ch 5) says that the unitary irreducible representation of these commutation relations is unique. Thus there must exist some invertible unitary operator ∇ such that

$$\beta'^i=\nabla(A)\beta^i\nabla(A)^{-1} \tag{4.9}$$

where ∇ is an infinite-dimensional matrix depending on the symplectic matrix A. We will call the representation

$$\nabla:A\to\nabla(A) \tag{4.10}$$

the 'symplectic spinor' representation of Sp(n).

It is interesting to note the interchange of roles of commuting and anticommuting variables in the construction of symplectic spinors. Here, the commuting variables used to define orthogonal spinors are replaced by anticommuting variables in the definition of symplectic spinors, while, in the approach of Thierry-Mieg (1984), the anticommuting variables used to construct the orthogonal spinors are replaced by commuting variables in the definition of the symplectic spinors. Following the discussion of the previous section and the results of Thierry-Mieg (1984), these symplectic spinors can be thought of as (infinite-dimensional) representations of O($-n$), or 'negative-dimensional spinors'. In each construction, we see that the passage from positive to negative dimensions involves an interchange of commuting and Grassmann variables.

In fact, the defining commutation relations (4.7) are the same as those arrived at by Čvitanović and Kennedy (1982) in the 'spinster' construction in which they considered the possibility of taking the usual γ matrix entries to be Grassmann valued. Using Penrose's diagrammatic methods, they showed that, in n dimensions, spinor traces can be reduced to sums of terms involving Fierz, $3-j$ and $6-j$ coefficients for antisymmetric representations of SO(n). Furthermore, they found that 'spinster' traces can be reduced to similar sums for symmetric tensor representations of Sp(n), and that the Fierz, $3-j$ and $6-j$ coefficients continue under $n \to -n$ from the spinor to the spinster case. Hence, the Sp(n) representation based on (4.7) can indeed by interpreted as a spinor representation of SO(n).

5. Positive- and negative-dimensional harmonic oscillator

In this section we consider the group theoretical approach to the spectral problem of the isotropic harmonic oscillator in positive and negative dimensions. In the positive-dimensional system (dim $=2n$), it is useful to define the usual bosonic Fock space with operators a_i, a_j^+ such that

$$[a_i, a_j^+] = \delta_{ij} \tag{5.1}$$

in terms of which the Hamiltonian is

$$H = a_i^+ \delta^{ij} a_j + n. \tag{5.2}$$

Now we can also represent the generators of SU($2n$) in the Cartan form using these Fock operators:

Cartan subalgebra generators: $H_i = a_i^+ a_i$ (no sum over i) $\qquad i = 1, \ldots, 2n-1 \qquad$ (5.3)

step operators for the positive roots $(e_i - e_j)$: $E(e_i - e_j) = a_i^+ a_j \qquad (i<j)$. $\qquad$ (5.4)

It is straightforward to check that these satisfy the required commutation relations

$$[H_k, E(e_i - e_j)] = \delta_{ki} E(e_k - e_j) - \delta_{kj} E(e_i - e_k)$$

$$[E(e_i - e_j), E(-e_i + e_j)] = H_i - H_j.$$

The trick of writing the Lie algebra generators in terms of Fock operators is originally due to Schwinger (1965). It is now easy to see that SU($2n$) is the (maximal) symmetry group of the harmonic oscillator system, since

$$[H, H_i] = 0 \qquad \forall i$$

and

$$[H, E(e_i - e_j)] = 0 \qquad \forall i, j.$$

Thus the eigenstates will correspond to irreducible representations of SU($2n$). In the standard Fock space construction, the vacuum $|0\rangle$ is annihilated by all the a_i, and hence is annihilated by all the SU($2n$) generators (5.3) and (5.4). The Fock state

$$|(m_1, \ldots, m_{2n})\rangle = \prod_{i=1}^{2n} (a_i^+)^{m_i} |0\rangle$$

has energy eigenvalue $n+m$ where $m = \Sigma_{i=1}^{2n} m_i$. The multiplicity is the number of ways of choosing the m_i such that $m = \Sigma_{i=1}^{2n} m_i$, which is $\binom{2n+m-1}{m}$ (the number of ways of choosing m objects from $2n$ with repetitions allowed). This agrees with the fact that since all the a_i^+ commute with each other, the Fock states with fixed energy $n+m$ form a totally symmetric irreducible representation of SU($2n$) which has dimension $\binom{2n+m-1}{m}$.

Now in the $2n$-dimensional Grassmann harmonic oscillator system it is also useful to define a 'Fock' space (Finkelstein and Villasante 1986, Dunne and Halliday 1988) using the oscillators α_i, α_j^+ such that

$$\{\alpha_i, \alpha_j^+\} = \varepsilon_{ij} \tag{5.5}$$

in terms of which the Hamiltonian is

$$\begin{aligned} H &= \alpha_i^+ \varepsilon^{ij} \alpha_j - n \\ &= \alpha_i^+ \alpha^i - n \end{aligned} \tag{5.6}$$

where ε^{ij} is used to raise the indices. It is remarkable (Gilmore 1974) that we can also use these Grassmann Fock operators to represent the SU($2n$) generators in Cartan form:

Cartan subalgebra generators: $H_i = \alpha_i^+ \alpha^i$ (no sum over i) $\qquad i = 1, \ldots, 2n-1$ (5.7)

step operators for positive roots: $E(e_i - e_j) = \alpha_i^+ \alpha^j \qquad i < j.$ (5.8)

Once again, SU($2n$) is the maximal symmetry group of the system since

$$[H, H_i] = 0 \qquad \forall i$$

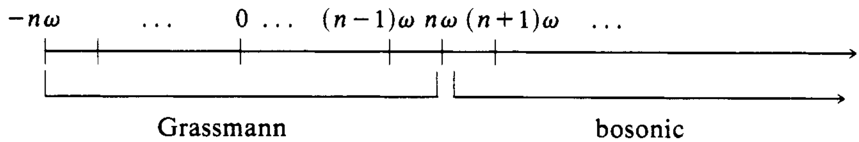
$$[H, E(e_i - e_j)] = 0 \qquad \forall i, j.$$

The vacuum $|0\rangle$ is annihilated by all the α_i, and so is SU($2n$) invariant. The Fock states are formed as

$$|(m_1, \ldots, m_{2n})\rangle = \prod_{i=1}^{2n} (\alpha_i^+)^{m_i} |0\rangle \tag{5.9}$$

but now, because of the Grassmann nature of the oscillators, each m_i can only be 0 or 1. This state has energy eigenvalue $-n + m$ (where m is defined as before), and the multiplicity is $\binom{2n}{m}$ (the number of ways of choosing m objects from $2n$ with *no* repetitions). This agrees with the fact that since the α^+ all anticommute with each other, the Fock states with fixed energy $-n + m$ form a totally antisymmetric irreducible representation of SU($2n$) with dimension $\binom{2n}{m}$.

Now let us compare the spectra of the $2n$-dimensional bosonic HO and of the $2n$-dimensional Grassmann HO (Dunne and Halliday 1988):



Grassmann: $E_m = -n\omega + m\omega$

$$\text{multiplicity} = \binom{2n}{m} \tag{5.10a}$$

bosonic: $E_m = n\omega + m\omega$

$$\text{multiplicity} = \binom{2n + m - 1}{m}. \tag{5.10b}$$

These spectra are both discrete, with energies quantised in units of ω. Notice that the Grassmann multiplicity function $\binom{2n}{m}$ vanishes for $m < 0$ and for $m > 2n$, and so the spectrum is finite. However, while $\binom{2n+m-1}{m}$ vanishes for $m < 0$, there is no upper limit on m, and hence the bosonic spectrum is semi-infinite (bounded below but not above). In fact, these spectra continue into one another under the analytic continuation $n \to -n$, as is clear from (5.10) if we note that

$$\binom{-2n}{m} = (-1)^m \binom{2n+m-1}{m}.$$

This somewhat mysterious result follows immediately from the results of this paper where it has been shown that Grassmann representations of SU($2n$) behave like the (transposed) bosonic representations of SU($2n$) with n continued to $-n$. The $(-1)^m$ factor appearing in the continuation of the multiplicities (compare with equation (3.1)) appears naturally in the Grassmann quantum theory due to the fact that the Grassmann Fock states (5.9) have indefinite normalisation.

To see this explicitly, consider obtaining the spectra from the respective Green functions by the following procedure. The $2n$-dimensional bosonic HO Green function is

$$G(x, t; x', t')$$

$$= \left(\frac{m\omega}{2\pi \sin \omega T}\right)^n \exp\left(\frac{-m\omega}{2 \sin \omega T}[(x^2+x'^2)\cos \omega T - 2x \cdot x']\right) \tag{5.11a}$$

$$= \sum_m \varphi_m(x)\varphi_m^*(x') \exp(-\mathrm{i}E_m T) \tag{5.11b}$$

where $T \equiv (t'-t)$ and $\{\varphi_m\}$ is a complete set of orthonormal energy eigenstates. Then

$$\int \mathrm{d}^{2n}x G(x, t; x, t') = \sum_m \text{multiplicity}(E_m) \exp(-\mathrm{i}E_m T)$$

from (5.11b). But from (5.11a) this equals

$$[-2(1-\cos \omega T)]^{-n} = \sum_m \binom{2n+m-1}{m} \exp[-\mathrm{i}(m+n)\omega T]$$

which gives the spectrum (5.10b). For the $2n$-dimensional Grassmann HO system, the Green function is (Dunne and Halliday 1987)

$$G(\Theta, t; \Theta', t')$$

$$= \left(\frac{m\omega}{2\pi \sin \omega T}\right)^{-n} \exp\left(\frac{-m\omega}{2 \sin \omega T}[(\Theta^2+\Theta'^2)\cos \omega T - 2\Theta \cdot \Theta']\right) \tag{5.12a}$$

$$= \sum_m \psi_m(\Theta)\bar{\psi}_m(\Theta') \exp(-\mathrm{i}E_m T) \tag{5.12b}$$

where now the energy eigenstates $\psi_m(\Theta)$ are normalised using the (indefinite) Grassmann inner product

$$\int \mathrm{d}^{2n}\Theta \psi_m(\Theta)\bar{\psi}_k(\Theta) = (-1)^m \delta_{mk}.$$

Then

$$\int \mathrm{d}^{2n}\Theta G(\Theta, t; \Theta, t') = \sum_m (-1)^m \text{multiplicity}(E_m) \exp(-\mathrm{i}E_m T)$$

from (5.12*b*). But from the explicit expression for the Green function we find that this equals

$$[-2(1-\cos\omega T)]^n=\exp(\mathrm{i}n\,\omega T)\sum_{n=0}^{2n}(-1)^m\binom{2n}{m}\exp(-\mathrm{i}m\omega T)$$

which gives the spectrum (5.10*a*), and illustrates clearly the origin of the $(-1)^m$ factor in the multiplicity.

We conclude that, while SU($2n$) is the symmetry group of the $2n$-dimensional bosonic harmonic oscillator, SU($-2n$) (as defined in § 3) is the symmetry group of the $2n$-dimensional Grassmann harmonic oscillator.

It is interesting to note how the two Fock spaces decompose into irreducible representations for these two systems. In the bosonic system, the Fock space is infinite dimensional and reduces into a direct sum of all the totally symmetric irreducible representations of SU($2n$):

$$\sum_{m=0}^{\infty}\binom{2n+m-1}{m}=\infty.$$

In the Grassmann system, the Fock space is finite dimensional (dimension 2^{2n}), and we can see explicitly how it reduces into a direct sum of all the totally antisymmetric irreducible representations of SU($2n$):

$$\sum_{m=0}^{\infty}\binom{2n}{m}=2^{2n}.$$

6. Negative-dimensional angular momentum

The angular momentum generators

$$L_{ij}=x_ip_j-x_jp_i \qquad i,j=1,\ldots,N \tag{6.1}$$

where $p_j=-\mathrm{i}\partial/\partial x_j$ generate SO(N) transformations. This realisation of the SO(N) generators as differential operators bridges the gap between Weyl's theory of invariants and the Lie-Cartan infinitesimal formulation of Lie groups and algebras. In the latter language, the operators (6.1) generate SO(N) transformations because

$$x_k\to x'_k=\exp(\tfrac{1}{2}\mathrm{i}b^{ij}L_{ij})x_k \tag{6.2}$$

(where $b^{ij}=-b^{ji}$ are infinitesimal real parameters) leaves the orthogonal inner product $\boldsymbol{x}^2=x_k\delta^{kl}x_l$ invariant. Furthermore, we can verify that the L_{ij} satisfy the SO(N) algebra commutation relations

$$[L_{ij},L_{kl}]=\mathrm{i}(\delta_{ik}L_{jl}-\delta_{il}L_{jk}-\delta_{jk}L_{il}+\delta_{jl}L_{ik}). \tag{6.3}$$

The quadratic Casimir of SO(N) is

$$\begin{aligned}C_2&=\tfrac{1}{2}\delta^{ij}\delta^{kl}L_{ik}L_{jl}\\&=\boldsymbol{x}^2\boldsymbol{p}^2+\mathrm{i}(N-2)\boldsymbol{x}\cdot\boldsymbol{p}-(\boldsymbol{x}\cdot\boldsymbol{p})^2.\end{aligned} \tag{6.4}$$

In an irreducible representation labelled by highest weight $\boldsymbol{\lambda}$,

$$C_2=\boldsymbol{\lambda}^2+\boldsymbol{\lambda}\cdot\sum_{\alpha>0}\boldsymbol{\alpha} \tag{6.5}$$

where $\alpha>0$ are the positive roots of $SO(N)$. For $SO(2n)$ the positive roots are $e_i \pm e_j (i<j;\ i=1,\ldots,n-1)$, while for $SO(2n+1)$ the positive roots are $e_i \pm e_j, (i<j;\ i=1,\ldots,n-1)$, and $e_i (i=1,\ldots,n)$. We are usually interested in the angular momentum of systems with commuting coordinates, so we are interested in the eigenvalues of C_2 in totally symmetric irreducible representations. Such an irreducible representation has highest weight

$$\boldsymbol{\lambda}=(l,0,\ldots,0). \tag{6.6}$$

Then, for $SO(N)$ we have

$$C_2=l(l+N-2). \tag{6.7}$$

The multiplicity of this eigenvalue is just the dimension of the corresponding totally symmetric irreducible representation. From Weyl's dimension formula

$$\dim(\boldsymbol{\lambda})=\left(\prod_{\alpha>0}(\boldsymbol{\delta}+\boldsymbol{\lambda})\cdot\boldsymbol{\alpha}\right)\left(\prod_{\alpha>0}\boldsymbol{\delta}\cdot\boldsymbol{\alpha}\right)^{-1} \tag{6.8}$$

(where $\boldsymbol{\delta}\equiv\frac{1}{2}\Sigma_{\alpha>0}\,\boldsymbol{\alpha}$), we find that

$$\dim(l,0,\ldots,0)=\left(\frac{2l+N-2}{N-2}\right)\binom{l+N-3}{l}. \tag{6.9}$$

When $N=3$ we recover the familiar case of angular momentum in three dimensions, for which

$$L^2\equiv C_2=l(l+1)$$

with multiplicity $2l+1$.

Motivated by the results of §§ 2 and 3, we would expect the above construction on a (symplectic) Grassmann space to yield 'negative-dimensional' angular momentum. In a $2n$-dimensional Grassmann space, the generators $L_{\alpha\beta}$ of infinitesimal transformations

$$\Theta_\gamma\to\exp(\tfrac{1}{2}\mathrm{i}b^{\alpha\beta}L_{\alpha\beta})\Theta_\gamma \tag{6.10}$$

(where $b^{\alpha\beta}=b^{\beta\alpha}$ are infinitesimal real parameters) which leave invariant the bilinear form $\boldsymbol{\Theta}^2\equiv\Theta_\alpha\varepsilon^{\alpha\beta}\Theta_\beta$ may be represented as

$$L_{\alpha\beta}=\Theta_\alpha\Pi_\beta+\Theta_\beta\Pi_\alpha \qquad \alpha,\beta=1,\ldots,2n \tag{6.11}$$

where

$$\Pi_\beta=\mathrm{i}\varepsilon_{\beta\gamma}\partial/\partial\Theta_\gamma.$$

Notice that $L_{\alpha\beta}$ is now symmetric in the α and β indices, so there are $n(2n+1)$ such generators. They satisfy the commutation relations

$$[L_{\alpha\beta},L_{\gamma\delta}]=\mathrm{i}(\varepsilon_{\gamma\beta}L_{\alpha\delta}+\varepsilon_{\gamma\alpha}L_{\beta\delta}+\varepsilon_{\delta\alpha}L_{\gamma\beta}+\varepsilon_{\delta\beta}L_{\gamma\alpha}) \tag{6.12}$$

which are the commutation relations for the generators of $Sp(2n)$. (It is interesting to note that a very similar construction has been used for $Sp(2)$ by Delbourgo (1988) in a formulation in which spin is realised in terms of Grassmann angular momentum.) The quadratic Casimir is

$$\begin{aligned}C_2&=\tfrac{1}{2}\varepsilon^{\alpha\beta}\varepsilon^{\gamma\delta}L_{\alpha\gamma}L_{\beta\delta}\\&=-\boldsymbol{\Theta}^2\boldsymbol{\Pi}^2-\mathrm{i}(2n+2)\boldsymbol{\Theta}\cdot\boldsymbol{\Pi}+(\boldsymbol{\Theta}\cdot\boldsymbol{\Pi})^2.\end{aligned} \tag{6.13}$$

In the Grassmann system we are interested in the angular momentum of a system with anticommuting coordinates, which means that we are looking for the eigenvalues of C_2 in totally antisymmetric irreducible representations of Sp$(2n)$. Such a representation has highest weight

$$\boldsymbol{\lambda}^{(l)}=(\underbrace{1,1,\ldots,1}_{l\ \text{times}},0,0,\ldots,0) \tag{6.14}$$

(note that when $l>n$, $\boldsymbol{\lambda}^{(l)}\equiv\boldsymbol{\lambda}^{(2n-l)}$). Sp$(2n)$ has positive roots $e_i\pm e_j$ $(i<j;\ i=1,\ldots,n-1)$ and $2e_i (i=1,\ldots,n)$. Hence, using (6.5), we have

$$C_2=-l(l-2n-2). \tag{6.15}$$

The multiplicity of this eigenvalue is just the dimension of the corresponding totally antisymmetric irreducible representation of Sp$(2n)$:

$$\dim(\boldsymbol{\lambda}^{(l)})=\left(\frac{n-l+1}{n+1}\right)\binom{2n+2}{l}. \tag{6.16}$$

Evaluating the bosonic results (6.7) and (6.9) at $N=-2n$ gives

$$C_2^{\mathrm{SO}(N)}(\text{boson})|_{N=-2n}=l(l-2n-2) \tag{6.17}$$

$$\begin{aligned}\dim((l,0,\ldots,0);\mathrm{SO}(N);\text{boson})|_{N=-2n}&=\left(\frac{n-1+1}{n+1}\right)\binom{l-2n-3}{l}\\&=(-1)^l\left(\frac{n-l+1}{n+l}\right)\binom{2n+2}{l}.\end{aligned} \tag{6.18}$$

Thus comparing (6.15) and (6.16) with (6.17) and (6.18) we find that the Grassmann angular momentum is naturally identified with negative-dimensional bosonic angular momentum. Once again, the $(-1)^l$ factor in the multiplicity is due to the indefinite normalisation of the Grassmann states. This result is an *algebraic* version of the group theoretical results (3.2) and (3.8).

Finally, we comment that the Grassmann spectral problem can be formulated as a finite-dimensional matrix problem (cf the 'operator' analysis of negative-dimensional quantum mechanics in Dunne and Halliday (1988)). For example, from (6.13) with $n=1$,

$$C_2=3\left(\Theta_1\frac{\partial}{\partial\Theta_1}+\Theta_2\frac{\partial}{\partial\Theta_2}\right)-6\Theta_1\Theta_2\frac{\partial^2}{\partial\Theta_2\partial\Theta_1}$$

which has a 4×4 matrix representation

$$C_2=\begin{pmatrix}0&&&\\&3&&\\&&3&\\&&&0\end{pmatrix}.$$

This clearly has eigenvalues 0 and 3, each with multiplicity 2. From (6.15) and (6.16) with $n=1$ (recalling that $\boldsymbol{\lambda}^{(l)}\equiv\boldsymbol{\lambda}^{(2n-l)}$ for $l>n$) we find

$l=0,2\rightarrow C_2=0$ each with multiplicity 1

$l=1\rightarrow C_2=3$ with multiplicity 2.

These agree with the matrix results above.

7. Conclusion and discussion

In this paper we have shown that the identification of Grassmann variables with 'negative dimensions' extends to the representation theory of the classical Lie groups. This identification leads to constructive definitions of negative-dimensional groups, whose irreducible characters are related to those of positive-dimensional groups in a way that naturally incorporates some remarkable identities concerning the analytic continuation of group invariants. Applying these results to the quantisation of Grassmann systems, we found that SU($-2n$) and SO($-2n$) are indeed the symmetry groups of the Grassmann harmonic oscillator and of Grassmann angular momentum, respectively.

These ideas suggest several intriguing generalisations. Firstly, while the relationship between Grassmann and bosonic tensor (and spinor) representations of the classical groups is quite straightforward in the Weyl framework discussed in § 2, it would be interesting to translate this directly into Cartan's (perhaps more familiar) algebraic framework. Also, if the negative-dimensional symmetry of the group characters could be incorporated directly into Weyl's character formula (when expressed in terms of roots and weights, rather than in the determinantal form of the appendix), then we may well find a further generalisation of these results to the exceptional groups, and even to infinite-dimensional Lie groups. Several things suggest that this latter generalisation should be meaningful. Firstly, Weyl's character formula itself has a natural and elegant extension to infinite-dimensional algebras (Kac 1985). Secondly, a simple construction of the Virasoro algebra in terms of bilinears of Grassmann oscillators:

$$L_m = \tfrac{1}{2}\colon \sum_{m=-\infty}^{\infty} \alpha^r_{m-n}\alpha^s_n \varepsilon_{rs} \colon$$

where

$$\{\alpha^r_m, \alpha^s_n\} = m\delta_{m,-n}\varepsilon^{rs} \qquad r, s = 1, \dots D$$

instead of the usual bosonic oscillators (see, for example, Goddard and Olive 1986) leads to the commutation relations

$$[L_m, L_n] = (m-n)L_{m+n} - \tfrac{1}{12}Dm(m^2-1)\delta_{m,-n}$$

which has a 'negative-dimensional' central extension term. (This is exactly analogous to the ghost systems in bosonic string theory.) Work is in progress to investigate a similar negative-dimensional construction for Kac–Moody algebras.

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Appendix. Weyl's character formula

In this appendix, we express the result (2.13) in an explicit determinantal form. Weyl (1939) gave an explicit character formula (apparently originally due to Shur) for the irreducible representations of GL(n):

$$\varphi_{(\lambda_1,\ldots,\lambda_k)}(A)=\begin{vmatrix} p_{\lambda_1} & p_{\lambda_1+1} & \cdots & p_{\lambda_1+k-1} \\ p_{\lambda_2-1} & p_{\lambda_2} & & p_{\lambda_2+k-2} \\ \vdots & \vdots & & \vdots \\ p_{\lambda_k-k+1} & p_{\lambda_k-k+2} & \cdots & p_{\lambda_k} \end{vmatrix} \tag{A1}$$

where p_j is the coefficient of z^j in the expansion of $1/\det(1-zA)$. This makes use of a determinantal identity first given by Jacobi. Now let the *same* Young tableau λ be specified by its column lengths $(\tilde{\lambda}_1,\ldots,\tilde{\lambda}_m)$ rather than by its row lengths $(\lambda_1,\ldots,\lambda_k)$. Then, using another determinantal identity due to Naegelsbach (see Littlewood 1950, ch VI and references therein), we can write this same character as

$$\begin{vmatrix} q_{\tilde{\lambda}_1} & q_{\tilde{\lambda}_1+1} & \cdots & q_{\tilde{\lambda}_1+m-1} \\ q_{\tilde{\lambda}_2-1} & q_{\tilde{\lambda}_2} & & q_{\tilde{\lambda}_2+m-2} \\ \vdots & \vdots & & \vdots \\ q_{\tilde{\lambda}_m-m+1} & q_{\tilde{\lambda}_m-m+2} & & q_{\tilde{\lambda}_m} \end{vmatrix}$$

where q_j is the coefficient of $(-z)^j$ in the expansion of $\det(1-zA)$. (Notice that these identities are quire remarkable, as in general they relate determinants of different degree.) We can use this result, combined with (2.13), to give a formula analogous to Weyl's character (A1) for Grassmann tensorial representation theory:

$$\varphi_\lambda(A;\,\text{Grassmann})=\begin{vmatrix} q_{\lambda_1} & q_{\lambda_2+1} & \cdots & q_{\lambda_1+k-1} \\ q_{\lambda_2-1} & q_{\lambda_2} & & q_{\lambda_2+k-2} \\ \vdots & \vdots & & \vdots \\ q_{\lambda_k-k+1} & q_{\lambda_k-k+2} & \cdots & q_{\lambda_k} \end{vmatrix} \tag{A2}$$

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